# CONCERNING THE NEGATIVE PART OF THE SPECTRUM OF ONE-DIMENSIONAL AND MULTI-DIMENSIONAL DIFFERENTIAL OPERATORS ON VECTOR-FUNCTIONS

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# CONCERNING THE NEGATIVE PART OF THE SPECTRUM OF ONE-DIMENSIONAL AND MULTI-DIMENSIONAL DIFFERENTIAL OPERATORS ON VECTOR-FUNCTIONS†

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The present note is devoted to an extension of theorems of note [1a], that supplements earlier obtained results [1b] concerning the spectrum of single-dimensional and multi-dimensional differential operators on vector-functions. /421*

Let $\vec{\mathscr{L}}_2(0,\infty)$ be a Hilbert space of vector-functions $y(x) = \{y_k\}_{k=1}^{m}$ $(m<\infty)$ with scalar product

$$(y, z) = \int_0^\infty \sum_{k=1}^{m} y_k(x)\overline{z_k(x)}\,dx,$$

and $l[y]$ be a differential operation of the form

$$l[y] = (-1)^n y^{(2n)} + Q(x)y \quad (0 \leqslant x < \infty), \tag{1}$$

where $Q(x)$ is a Hermite matrix-function of the m-th order. The least and, respectively, the greatest eigenvalue of matrix $Q(x)$ let us designate by $\mu(x)$ and $\nu(x)$. By $\tilde{L}$ let us designate any self-adjoint expansion of an operator with minimal area of determination generated in $\vec{\mathscr{L}}_2(0,\infty)$ by operation (1). The negative part of any function $f(x)$ let us designate by $f^*(x)$, such that $f^*(x) = \min\{0, f(x)\}$.

The use of lemma 1 of note [1a], where one must replace functional $\Phi_\varepsilon[y]$ by functional

†Presented by Academician S. N. Bernshteyn, October 24, 1957.
*Numbers in right hand margin indicate pagination of foreign text.

$$\Phi_\varepsilon[y] = \int_0^\infty \sum_{k=1}^m |y_k^{(n)}(x)|^2 dx + \int_0^\infty \sum_{j,k=1}^m Q_{jk}(x) y_j(x) \overline{y_k(x)} dx + \\ + \varepsilon \int_0^\infty \sum_{k=1}^m |y_k(x)|^2 dx,$$

leads to the following results.

Theorem 1. If in the case of any $\delta>0$ is fulfilled the inequality

$$\int_{M_\delta} |\mu^*(x)| dx < \infty,$$

where $M_\delta$ is the set of values x, for which $|\mu^*(x)| \geq \delta$, then the negative part of the spectrum of operator $\mathfrak{L}$ is bounded below and discrete.

Assuming, further, /422

$$\alpha_n = \frac{(2n-1)!!}{2^n}, \quad A_n = (2n-1)^{-1/2} \left[ \sum_{k=1}^n \frac{(-1)^{k-1} C_{n-1}^{k-1}}{2n-k} \right]^{-1} (n-1)!,$$

$$B_n^2 = \frac{n(4n^2-1)}{3 \cdot 4^{n-1}} \sum_{k=1}^n \frac{1}{2k-1} \sum_{k=0}^{2n-2} \frac{(-1)^k C_{2n-2}^k}{4n-3-k} \left[ \sum_{k=1}^n \frac{(-1)^{k-1} C_{n-1}^{k-1}}{2n-k} \right]^{-2},$$

let us mention the following two theorems.

Theorem 2. The negative part of the spectrum of operator $\mathfrak{L}$ consists of a finite number of eigenvalues, if one of the following conditions is fulfilled:

1. $\mu(x) \geq -\alpha_n^2 x^{-2n}$ in the case of large x.
2. In the case of any $\delta>0$

$$\int_{M_\delta} x^{2n-1} |\mu^*(x)| dx < \infty,$$

where $M_\delta$ is the set of values x for which

$$|\mu^*(x)| \geqslant (\alpha_n^2 - \delta) x^{-2n}.$$

3. In the case of some $p \geq 1$

$$\int_0^\infty x^{2np-1}|\mu^*(x)|^p\,dx < \infty.$$

Theorem 3. The negative part of the spectrum of operator $L$ is an infinite set if one of the following conditions is fulfilled:

1. In case of some $\delta > 0$ and large x

$$\nu(x) < -(\alpha_n^2 + \delta)\,x^{-2n}.$$

2. $\nu(x) \leq 0$ in case of large values of x and

$$\lim_{\rho\to\infty}\inf \rho^{2n-1}\int_\rho^\infty |\nu(x)|\,dx > A_n^2.$$

3. $\nu(x) \leq -\alpha_n^2 x^{-2n}$ in the case of large x and

$$\lim_{\rho\to\infty}\inf \ln\rho \int_\rho^\infty x^{2n-1}|\nu(x) + \alpha_n^2 x^{-2n}|\,dx > B_n^2.$$

4.

$$\int_0^\infty \nu(x)\,dx = -\infty.$$

In conditions 2 and 3 one can replace $\lim\limits_{\rho\to\infty}\inf$ by $\lim\limits_{\rho_k\to\infty}$.

Theorems 1-3 are connected with the oscillation properties of the system of differential equations

$$(-1)^n y^{(2n)} + Q(x)\,y = \lambda y \quad (\lambda \leqslant 0),$$

that were studied by Sternberg [2] in the case of $n = 1$. Conditions 1 of theorem 2 and 1 of theorem 3 give an extension of the well-known theorem of Knezer concerning the oscillation of solutions of a differential equation of the second order. In the case of $n = 1$ one can obtain the following refinement of condition 1 of theorem 3 that for the case of a differential equation of the /423 second order was given by Hille [3] (see also [4]).

Theorem 4. If in the case of some $\delta>0$ and some natural r for all sufficiently large values of x occurs the inequality

$$v(x) < -\frac{1}{4x^2} - \frac{1}{4x^2 \ln^2 x} - \dots - \frac{1+\delta}{4x^2 \ln^2 x \dots \ln_r^2 x},$$

where $\ln_k x = \ln \ln_{k-1} x$, then the negative part of the spectrum of operator $\tilde{L}$ consists of a finite number of eigenvalues.

The presented results are partially extended to multi-dimensional differential operations on vector-functions of the form

$$l[\mathbf{u}] = -\Delta \mathbf{u} + Q(P)\mathbf{u}, \tag{2}$$

where P is a point of n-dimensional Euclidean space $\mathscr{E}$; Q(P) is a Hermitian matrix-function of the m-th order determined in all $\mathscr{E}$.

Operation (2) generates in Hilbert space $\vec{\mathscr{L}}_2(\mathscr{E})$ of vector-functions $u(P) = \{u_k(P)\}_{k=1}^m$ with scalar product

$$(\mathbf{u}, \mathbf{v}) = \int_{\mathscr{E}} \sum_{k=1}^{m} u_k(P) \overline{v_k(P)}\, d\omega_P$$

some differential operator L with a minimum area of determination.

Let $\mu(P)$ be the least eigenvalue of matrix Q(P) and $\mu^*(P) = \min\{0, \mu(P)\}$. Let us present, for example, a formulation of the theorem that corresponds to theorem 1, and let us prove it for m = 1, n = 2 (in this case $\mu(P) = Q(P)$).

Theorem 5. If in the case of $\delta>0$ the integral

$$\int_0^\infty |\mu_\delta^*(P)|\, dr,$$

where

$$\mu_\delta^*(P) = \begin{cases} \mu^*(P), & |\mu^*(P)| > \delta, \\ 0, & |\mu^*(P)| \leqslant \delta, \end{cases}$$

converges uniformly along angular coordinates, then:

1) operator L with a minimum area of determination (see [1c]) is self-adjoint;

2) the negative part of the spectrum of operator L is semi-bounded below and discrete (that is, consists of eigenvalues of finite multiplicity with a single possible limiting point $\lambda = 0$).

Proof. Converting the quadratic functional

$$\Phi_\varepsilon[u] = \iint_{(\mathcal{E})} |\nabla u|^2 r\,dr\,d\varphi + \iint_{(\mathcal{E})} Q\,|u|^2 r\,vr\,d\varphi + \varepsilon\iint_{(\mathcal{E})} |u|^2 r\,dr\,d\varphi$$

to any finite function $u \in D_L$ with the help of the substitution of variables $u\sqrt{r} = v$, let us obtain

$$\Phi_\varepsilon[u] = \iint_{(\mathcal{E})} |\nabla v|^2\,dr\,d\varphi + \iint_{(\mathcal{E})} \left[Q(r,\varphi) + \frac{1}{4r^2}\right]|v|^2\,dr\,d\varphi + \varepsilon\iint_{(\mathcal{E})} |v|^2\,dr\,d\varphi.$$

In the case of randomly assigned $\varepsilon$ ($0<\varepsilon<1$) let us select a /424
number N such that there would be

$$\int_N^\infty |Q^*(r,\varphi)|\,dr < \frac{\varepsilon}{4},$$

and let us show that the functional

$$\Phi_\varepsilon[u] = \int_0^{2\pi} d\varphi \int_N^\infty \left\{|\nabla v|^2 + \left[Q^*(r,\varphi) + \frac{1}{4r^2} + \varepsilon\right]|v|^2\right\} dr$$

is nonnegative on any finite function $v \in D_L$ that equals zero in the range $r<N$.

From the Cauchy-Bunyakovsky inequality it follows that

$$\int_0^{2\pi}\int_N^\infty |\nabla v|^2 dr\,d\varphi \geqslant \int_0^{2\pi} d\varphi \int_N^\infty \left|\frac{\partial v}{\partial r}\right|^2 dr \geqslant \frac{1}{4}\left\{\int_0^{2\pi} |\hat{v}(\varphi)|^2 d\varphi\right\}^2,$$

where the function $v(r, \phi)$ is normalized by the condition

$$\int_0^{2\pi} d\varphi \int_N^\infty |v|^2\,dr = 1$$

and

$$\hat{v}(\varphi) = \max_{N \leqslant r < \infty} |v(r, \varphi)|.$$

Let us consider two cases separately:

1. $\int_0^{2\pi} |\hat{v}(\varphi)|^2 d\varphi \leqslant \varepsilon.$ 2. $\int_0^{2\pi} |\hat{v}(\varphi)|^2 d\varphi > \varepsilon.$

In the first case

$$\Phi_\varepsilon[u] \geqslant \int_0^{2\pi} \int_N^\infty Q^*(r, \varphi) |v|^2 dr\, d\varphi + \varepsilon \int_0^{2\pi} \int_N^\infty |v|^2 dr\, d\varphi \geqslant$$

$$\geqslant -\int_0^{2\pi} |\hat{v}(\varphi)|^2 \int_N^\infty |Q^*(r, \varphi)| dr\, d\varphi + \varepsilon \geqslant \varepsilon\left(1 - \frac{\varepsilon}{4}\right) > 0.$$

In the second case

$$\Phi_\varepsilon[u] \geqslant \frac{1}{4}\left\{\int_0^{2\pi} |\hat{v}(\varphi)|^2 d\varphi\right\}^2 + \int_0^{2\pi} \int_N^\infty Q^*(r, \varphi) |\hat{v}(\varphi)|^2 dr\, d\varphi,$$

such that

$$\Phi_\varepsilon[u] \geqslant \frac{1}{4} \int_0^{2\pi} |\hat{v}(\varphi)|^2 d\varphi \left[\int_0^{2\pi} |\hat{v}(\varphi)|^2 d\varphi - \varepsilon\right] > 0,$$

and inequality $\Phi_\varepsilon[u] \geq 0$ is established.

From the given inequality first of all follows [1d] the semi-boundedness of operator L below, and from that, according to a theorem of A. Ya. Povzner, the self-adjointness of operator L follows.

Further from this inequality on the basis of [1d] and of lemma 1 [1a] let us conclude that the negative part of the spectrum of operator L is discrete. The theorem is proven.

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## REFERENCES

1. I. M. Glasman, a) DAN, Vol. 118, No. 3 (1958); b) Uch. zap. Khar'kovsk. gos. ped. inst., Vol. 18, mathematical series (1956); c) Tr. Khar'kovsk. nolitekhn. inst., Vol. 5, series engineering-physics, in 1 (1955); d) Matem. sborn., Vol. 35(77), 2, (1954).

2. R. Sternberg, Duke Math. Journal, Vol. 19, 311 (1952).

3. E. Hille, Trans. Am. Math. Soc., Vol. 64, No. 234 (1948).

4. R. Bellman, Theory of Stability [Teoriya ustoychivosti], Foreign Literature Press, 1954.

5. A. Ya. Povzner, Mathematical Collection [Matem. sborn.], Vol. 32(74), 1 (1953).

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16. Abstract The article extends theorems of previous work by I.M. Glasman concerning the spectrum of one-dimensional and multi-dimensional differential operators on vector functions. Given a Hilbert space of vector fumctions and a differential operator of specific form, it is first shown that the negative part of the spectrum of any self-adjoint expansion of an operator L is discrete and bounded below (Theorem 1). Further specification of two separate sets of conditions causes the negative part of the spectrum of operator L (1) to consist of a finite number of eigenvalues (Th. 2) and (2) to be an infinite set (Th. 3). These 3 theorems are connected with the oscillation properties of certain differential equations studied by Sternberg [2]. Refinement of one condition of Theorem 3 applied to a differential equation of second order results in the negative part of the spectrum of operator L consisting of a finite number of eigenvalues (Th. 4). Then it is proven that for specific integral types operator L with a minimum area of determination is self-adjoint and (cont)

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the negative part of the spectrum of operator L is discrete and semi-bounded below.